Appendix 1: Creating closed paths on a sphere using apparent and true axial rotation

xz'y'' Decomposition

Let $(\alpha_0, \beta_0, \gamma_0)$ represent the sequential angles of rotation about the x, z', and y'' axes of the moving frame of the humerus to reach the initial orientation of the humerus, respectively. Likewise, let $(\alpha_n, \beta_n, \gamma_n)$ represent the angles necessary to reach the final orientation of the humerus. The following closed path can be constructed starting from the initial orientation of the humerus.

1. Undo initial axial orientation: $(\alpha_0, \beta_0, \gamma_0) \rightarrow (\alpha_0, \beta_0, 0)$
2. Go back to zero latitude: $(\alpha_0, \beta_0, 0) \rightarrow (\alpha_0, 0, 0)$
3. Go back to zero longitude and latitude: $(\alpha_0, 0, 0) \rightarrow (0, 0, 0)$
4. Go to final longitude: $(0, 0, 0) \rightarrow (\alpha_n, 0, 0)$
5. Go to final longitude and latitude: $(\alpha_n, 0, 0) \rightarrow (\alpha_n, \beta_n, 0)$
6. Add final axial orientation: $(\alpha_n, \beta_n, 0) \rightarrow (\alpha_n, \beta_n, \gamma_n)$
7. Go to initial orientation via the actual path traveled by the humerus (traversed backward): $(\alpha_n, \beta_n, \gamma_n) \rightarrow (\alpha_0, \beta_0, \gamma_0)$

Steps 3 and 4 could be combined without affecting the spherical area as follows: Go to final longitude from initial longitude along the zero-latitude line $(\alpha_0, 0, 0) \rightarrow (\alpha_n, 0, 0)$.
In this closed trajectory, apparent axial rotation is $-\gamma_0 + \gamma_n = \gamma_n - \gamma_0$. Let the true axial rotation (measured in Step 7) equal θ. Then the following holds: $|Spherical\ Area| = |\theta - (\gamma_n - \gamma_0)|$.

yx'y'' Decomposition

The rules of spherical geometry state that the difference in axial rotation between the sequence-specific trajectory and the true trajectory is equal to the enclosed area if there is no additional axial rotation. Since the yx'y'' decomposition imparts axial rotation when establishing the plane of elevation this amount must be considered.

Let $(\alpha_0, \beta_0, \gamma_0)$ represent the sequential angles of rotation about the y, x', and y'' axes of the moving frame of the humerus to reach the initial orientation of the humerus, respectively. Likewise, let $(\alpha_n, \beta_n, \gamma_n)$ represent the angles necessary to reach the final orientation of the humerus. The following closed path can be constructed from the initial orientation of the humerus.

1. Undo initial axial orientation: $(\alpha_0, \beta_0, \gamma_0) \rightarrow (\alpha_0, \beta_0, 0)$
2. Go back to zero latitude: $(\alpha_0, \beta_0, 0) \rightarrow (\alpha_0, 0, 0)$
3. Undo establishing initial plane of elevation: $(\alpha_0, 0, 0) \rightarrow (0, 0, 0)$
4. Establish final plane of elevation: $(0, 0, 0) \rightarrow (\alpha_n, 0, 0)$
5. Go to final longitude and latitude: $(\alpha_n, 0, 0) \rightarrow (\alpha_n, \beta_n, 0)$
6. Add final axial orientation: $(\alpha_n, \beta_n, 0) \rightarrow (\alpha_n, \beta_n, \gamma_n)$
7. Go to initial orientation via the actual path traveled by the humerus (traversed backward): $(\alpha_n, \beta_n, \gamma_n) \rightarrow (\alpha_0, \beta_0, \gamma_0)$

In this closed trajectory, apparent axial rotation is $-\gamma_0 + \gamma_n = \gamma_n - \gamma_0$. Let the true axial rotation (measured in Step 7) equal θ. Accounting for the fact that $\alpha_n - \alpha_0$ represents rotation about the longitudinal axis of the humerus while the plane of elevation is established then the following holds: $|Spherical\ Area| = |\theta - (\gamma_n - \gamma_0) - (\alpha_n - \alpha_0)|$.